MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 143
JULY 1978, PAGES 900–904

Further Evaluation of Howland Integrals

By Chih-Bing Ling

Abstract. The purpose of this paper is to further evaluate two Howland integrals to 25D when their index is an even integer.

In a previous paper [1], Ling and Lin evaluated the following two Howland integrals to 25D when their index k is an odd integer:

$$\begin{matrix} I_k \\ I_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k\,dw}{\sinh w \pm w} \qquad \begin{matrix} (k \geqslant 1) \\ (k \geqslant 3). \end{matrix} \tag{1}$$

Recently, the author encountered a need of highly precise values of these two integrals when their index is an even integer. This occurs in the evaluation of certain allied integrals of a similar nature. The computation incurs rapid loss of significant figures, although the desired accuracy is of a lower degree. In order to meet the need, the two integrals are further evaluated in this paper to 25D when their index is an even integer.

The following expansions shown in the previous paper hold for any index k, even or odd:

$$\begin{matrix} I_k \\ I_k^* \end{matrix} = 1 \mp \frac{q_2(k)}{2^{k+2}} + \frac{q_3(k)}{3^{k+3}} \mp \frac{q_4(k)}{4^{k+4}} + \cdots, \qquad \begin{matrix} (k \geqslant 1) \\ (k \geqslant 3), \end{matrix} \tag{2}$$

where, for $n \geqslant 0$,

$$\begin{aligned} q_{2n+1}(k) &= \sum_{m=0}^{n} (-1)^{n+m} \binom{2m+k}{k} \frac{(n+m)!}{(n-m)!} 2^{2m}(2n+1)^{2n-2m}, \\ q_{2n+2}(k) &= \sum_{m=0}^{n} (-1)^{n+m} \binom{2m+k+1}{k} \frac{(n+m+1)!}{(n-m)!} 2^{2m+1}(2n+2)^{2n-2m}. \end{aligned} \tag{3}$$

The series in (2) converges rather slowly when the index is a small integer. For instance, to attain an accuracy of 25D, 50 terms of the series are needed when $k \leqslant 20$; but only ten terms are needed when $k \geqslant 33$.

When the index is an even integer, the following two pairs of expansions are derived in a similar manner by using the method described in the previous paper, which is a modification of Plana's method:

Received August 15, 1977.
AMS (MOS) subject classifications (1970). Primary 41A30, 65A05.
Key words and phrases. Evaluation of Howland integrals.

$$I_{2k} = \frac{1}{(2k)!}\left[\frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(n\alpha)^{2k}\,\mathrm{Si}(n\pi)}{\sinh n\alpha + n\alpha}\right.$$
$$\left. + \tfrac{1}{2}\,\mathrm{Re}\sum_{m=1}^{\infty}\frac{z_m^{2k}\{E_1(a_m)\exp(a_m) + E^*(a_m)\exp(-a_m)\}}{\cosh^2(z_m/2)\sinh a_m}\right],$$

$$I_{2k} = \frac{1}{(2k)!}\left[\frac{\alpha}{\pi}\sum_{n=0}^{\infty}\frac{(n\alpha + \tfrac{1}{2}\alpha)^{2k}\,\mathrm{Si}(n\pi + \tfrac{1}{2}\pi)}{\sinh(n+\tfrac{1}{2})\alpha + (n+\tfrac{1}{2})\alpha}\right.$$
$$\left. + \tfrac{1}{2}\,\mathrm{Re}\sum_{m=1}^{\infty}\frac{z_m^{2k}\{E_1(a_m)\exp(a_m) - E^*(a_m)\exp(-a_m)\}}{\cosh^2(z_m/2)\cosh a_m}\right],$$

(4)

$$I^*_{2k} = \frac{1}{(2k)!}\left[\frac{\alpha}{\pi}\sum_{n=1}^{\infty}\frac{(n\alpha)^{2k}\,\mathrm{Si}(n\pi)}{\sinh n\alpha - n\alpha}\right.$$
$$\left. + \tfrac{1}{2}\,\mathrm{Re}\sum_{m=1}^{\infty}\frac{z_m^{*2k}\{E_1(a_m^*)\exp(a_m^*) + E^*(a_m^*)\exp(-a_m^*)\}}{\sinh^2(z_m^*/2)\sinh a_m^*}\right],$$

$$I^*_{2k} = \frac{1}{(2k)!}\left[\frac{\alpha}{\pi}\sum_{n=0}^{\infty}\frac{(n\alpha + \tfrac{1}{2}\alpha)^{2k}\,\mathrm{Si}(n\pi + \tfrac{1}{2}\pi)}{\sinh(n+\tfrac{1}{2})\alpha - (n+\tfrac{1}{2})\alpha}\right.$$
$$\left. + \tfrac{1}{2}\,\mathrm{Re}\sum_{m=1}^{\infty}\frac{z_m^{*2k}\{E_1(a_m^*)\exp(a_m^*) - E^*(a_m^*)\exp(-a_m^*)\}}{\sinh^2(z_m^*/2)\cosh a_m^*}\right],$$

where

(5) $$a_m = -\pi i z_m/\alpha, \qquad a_m^* = -\pi i z_m^*/\alpha.$$

The first pair is valid for $k \geqslant 1$ and the second pair for $k \geqslant 2$. α is a positive constant, which can be fixed to suit our convenience. z_m and z_m^* are the mth complex zeros of $(\sinh z \pm z)$, respectively, in the first quadrant of the z plane. Si is a sine integral and E_1 and E^* are exponential integrals [2] defined by

$$\mathrm{Si}(a) = \int_0^a \frac{\sin t}{t}\,dt,$$

(6) $$E_1(a) = \int_a^{\infty}\frac{e^{-t}}{t}\,dt,$$

$$E^*(a) = -\int_{-a}^{\infty}\frac{e^{-t}}{t}\,dt, \qquad (\mathrm{Re}[a] > 0).$$

The last integral is a Cauchy principal value. They are introduced into the expansions through the integrals:

$$\int_0^\infty \frac{\sin(\pi t/\alpha)\,dt}{n^2\alpha^2 - t^2} = -\frac{(-1)^n}{n\alpha}\,\mathrm{Si}(n\pi),$$

$$\int_0^\infty \frac{t\cos(\pi t/\alpha)\,dt}{(n\alpha + \tfrac{1}{2}\alpha)^2 - t^2} = (-1)^n\,\mathrm{Si}(n\pi + \tfrac{1}{2}\pi),$$

(7)
$$\int_0^\infty \frac{\sin(\pi t/\alpha)\,dt}{t^2 + a^2} = \frac{1}{2a}\left\{E_1\left(\frac{\pi a}{\alpha}\right)\exp\left(\frac{\pi a}{\alpha}\right) + E^*\left(\frac{\pi a}{\alpha}\right)\exp\left(-\frac{\pi a}{\alpha}\right)\right\},$$

$$\int_0^\infty \frac{t\cos(\pi t/\alpha)\,dt}{t^2 + a^2} = \tfrac{1}{2}\left\{E_1\left(\frac{\pi a}{\alpha}\right)\exp\left(\frac{\pi a}{\alpha}\right) - E^*\left(\frac{\pi a}{\alpha}\right)\exp\left(-\frac{\pi a}{\alpha}\right)\right\}.$$

Although the preceding expansions are more complicated than the analogous ones for an odd index, yet their properties are essentially alike. Each expansion consists of two series. Each first series converges more rapidly when α is large and each second series when α is small. In the computation, α is likewise taken as unity. The four first series involve $\mathrm{Si}(n\pi/2)$ and the four second series z_m, z_m^*, E_1 and E^*. The values of $\mathrm{Si}(n\pi/2)$ have been tabulated by Ling and Lin [3] to 25D for $n = 1(1)200$, together with a factor $2/\pi$. Further values can be generated easily whenever needed. The complex zeros z_m and z_m^* have been tabulated by Ling and Cheng [4] to 11D for both real and imaginary parts. Their accuracy can be improved readily by using the Newton-Raphson method. The following series are suitable for computing E_1 and E^* when $|\arg a| < \pi$:

(8)
$$E_1(a) = -\gamma - \ln a + e^{-a}\sum_{n=1}^{\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\frac{a^n}{n!},$$

$$E^*(a) = \gamma + \ln a + \sum_{n=1}^{\infty}\frac{a^n}{n(n!)},$$

where γ is Euler constant. Or, when $|a|$ is large, they are given by the following asymptotic series:

(9)
$$E_1(a) \sim \frac{e^{-a}}{a}\left(1 - \frac{1}{a} + \frac{2!}{a^2} - \frac{3!}{a^3} + \cdots\right),$$

$$E^*(a) \sim \frac{e^{a}}{a}\left(1 + \frac{1}{a} + \frac{2!}{a^2} + \frac{3!}{a^3} + \cdots\right),$$

so that

(10)
$$E_1(a)e^a + E^*(a)e^{-a} \sim \frac{2}{a}\left(1 + \frac{2!}{a^2} + \frac{4!}{a^4} + \cdots\right),$$

$$E_1(a)e^a - E^*(a)e^{-a} \sim -\frac{2}{a^2}\left(1 + \frac{3!}{a^2} + \frac{5!}{a^4} + \cdots\right).$$

Note that in the present computation the real part of a_m or a_m^* is always positive and greater than the imaginary part numerically.

TABLE 1

Howland integrals I_{2k} and I^*_{2k}

2k	I_{2k}	I^*_{2k}
2	0.76784 74391 33919 04735 95563	∞
4	0.88350 68065 08692 59048 39136	1.35329 41151 70484 00917 07709
6	0.95419 15618 26139 06398 92330	1.07672 97636 74217 12721 31355
8	0.98412 41801 48424 61430 92995	1.02053 76000 64959 33956 30213
10	0.99492 24398 53445 37390 51763	1.00578 48422 24523 50460 14460
12	0.99845 99579 47832 38304 10890	1.00164 49762 53520 38988 10881
14	0.99954 93055 62626 17685 89104	1.00046 58410 12174 41784 75036
16	0.99987 13214 26371 77772 61970	1.00013 08088 09455 33470 89835
18	0.99996 39030 69165 32295 73410	1.00003 63897 74380 02428 04253
20	0.99999 00058 51207 35354 88793	1.00001 00336 28030 38688 48032
22	0.99999 72607 78826 41420 23605	1.00000 27444 55935 02623 90687
24	0.99999 92552 82148 05839 47506	1.00000 07454 02213 09431 30333
26	0.99999 97988 78365 74311 66056	1.00000 02012 10025 40280 85053
28	0.99999 99459 88927 60932 42466	1.00000 00540 22369 86530 35487
30	0.99999 99855 65214 61417 58450	1.00000 00144 36216 21557 16743
32	0.99999 99961 58384 19634 14263	1.00000 00038 41795 57087 48030
34	0.99999 99989 81377 14198 99581	1.00000 00010 18645 28389 31179
36	0.99999 99997 30790 95789 84203	1.00000 00002 69211 82207 76456
38	0.99999 99999 29059 58456 74027	1.00000 00000 70940 75809 71445
40	0.99999 99999 81355 37962 27125	1.00000 00000 18644 66239 92477
42	0.99999 99999 95111 46854 25576	1.00000 00000 04888 53658 68928
44	0.99999 99999 98721 02338 73696	1.00000 00000 01278 97723 61257
46	0.99999 99999 99666 04495 19398	1.00000 00000 00333 95512 35557
48	0.99999 99999 99912 95851 94238	1.00000 00000 00087 04148 96852
50	0.99999 99999 99977 35145 08454	1.00000 00000 00022 64855 02501
52	0.99999 99999 99994 11581 80352	1.00000 00000 00005 88418 20962
54	0.99999 99999 99998 47344 33490	1.00000 00000 00001 52655 66667
56	0.99999 99999 99999 60448 30484	1.00000 00000 00000 39551 69535
58	0.99999 99999 99999 89765 13150	1.00000 00000 00000 10234 86852
60	0.99999 99999 99999 97354 54670	1.00000 00000 00000 02645 45330
62	0.99999 99999 99999 99316 95263	1.00000 00000 00000 00683 04737
64	0.99999 99999 99999 99823 81715	1.00000 00000 00000 00176 18285
66	0.99999 99999 99999 99954 59903	1.00000 00000 00000 00045 40097
68	0.99999 99999 99999 99988 31095	1.00000 00000 00000 00011 68905
70	0.99999 99999 99999 99996 99303	1.00000 00000 00000 00003 00697
72	0.99999 99999 99999 99999 22708	1.00000 00000 00000 00000 77292
74	0.99999 99999 99999 99999 80148	1.00000 00000 00000 00000 19852
76	0.99999 99999 99999 99999 94905	1.00000 00000 00000 00000 05095
78	0.99999 99999 99999 99999 98693	1.00000 00000 00000 00000 01307
80	0.99999 99999 99999 99999 99665	1.00000 00000 00000 00000 00335
82	0.99999 99999 99999 99999 99914	1.00000 00000 00000 00000 00086
84	0.99999 99999 99999 99999 99978	1.00000 00000 00000 00000 00022
86	0.99999 99999 99999 99999 99994	1.00000 00000 00000 00000 00006
88	0.99999 99999 99999 99999 99999	1.00000 00000 00000 00000 00001
90	1.00000 00000 00000 00000 00000	1.00000 00000 00000 00000 00000

To attain an accuracy of 25D for the two integrals with $\alpha = 1$, 190 terms in the first series and three terms in the second series are needed when $2k \leqslant 60$, or 130 terms in the first series and three terms in the second series when $2k \leqslant 30$. If the value of α is doubled, the number of terms needed in the first series is halved but that in the second series is doubled.

The resulting values of the two integrals are computed from (4) for $2k \leqslant 60$ and from (2) for $2k \geqslant 56$. The overlapped values are used for checking purposes. The values computed from each pair of expansions in (4) are in agreement as they ought to be. The following relations may be used as an additional check:

$$\sum_{k=1}^{\infty} k(1 - I_{2k}) = I_2 - \frac{1}{16}, \qquad \sum_{k=2}^{\infty} k(I_{2k}^* - 1) = \frac{17}{16}. \tag{11}$$

Comparison was made with Nelson's 18D values [5]. It revealed no discrepancy in Nelson's results.

The computation was carried out on an IBM 370 Computer with extended precision; and the 25D results for $2k$ from 2 to 90, inclusive, appear in Table 1.

In conclusion, it may be mentioned that, as an alternate method of evaluation, one might attempt to use the Gregory-Newton interpolation formula to evaluate the two integrals, because the values for odd index have been tabulated. However, it was found that by this formula adequate precision could not be obtained, particularly for small even index, owing to the behavior of the integrals.

Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

1. C. B. LING & J. LIN, "A new method of evaluation of Howland integrals," *Math. Comp.*, v. 25, 1971, pp. 331–337. MR 45 #4603.

2. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER & F. G. TRICOMI, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953, pp. 143–145. MR 15, 419.

3. C. B. LING & J. LIN, "A table of sine integral Si($n\pi/2$)," *Math. Comp.*, v. 25, 1971, RMT 21, p. 402.

4. C. B. LING & F. H. CHENG, "Stresses in a semi-infinite strip," (appendix), *Internat. J. Engrg. Sci.*, v. 5, 1967, p. 169.

5. C. W. NELSON, "New tables of Howland's and related integrals," *Math. Comp.*, v. 15, 1961, pp. 12–18. MR 22 #10203.